### 3.5 Dual Method for Solving Linear Programming Models

#### Standard Minimization Problems

In this section we will discuss a method that is used to solve a minimization problem when all of the problem constraints are written as greater than or equal to some number. Unlike the restrictions on the simplex method, the number can be positive, negative or even zero. The method we will use is the Dual Method. Before we get to the dual method, lets look at some standard minimization problems. (Notice that the constants can be negative or zero).

<table>
<thead>
<tr>
<th>Minimize: $C = 5x_1 + 2x_2$</th>
<th>Minimize: $Z = 3x_1 + 4x_2$</th>
<th>Minimize: $Z = 5x_1 - 3x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 + 3x_2 \geq 15$</td>
<td>$-2x_1 - 4x_2 \geq -24$</td>
<td>$2x_1 - 3x_2 \geq 15$</td>
</tr>
<tr>
<td>Subject To: $2x_1 + x_2 \geq 20$</td>
<td>$-3x_1 - 3x_2 \geq -21$</td>
<td>$4x_1 + 2x_2 \geq 20$</td>
</tr>
<tr>
<td>$x_1, x_2 \geq 0$</td>
<td>$4x_1 + 2x_2 \geq 20$</td>
<td>$x_1, x_2 \geq 0$</td>
</tr>
<tr>
<td>Subject To: $x_1, x_2 \geq 0$</td>
<td>$x_1, x_2 \geq 0$</td>
<td></td>
</tr>
</tbody>
</table>

#### Transpose of a Matrix

In order to apply the Dual Method, we must introduce the transpose of a matrix. Suppose a matrix $A$ is given. Then the transpose of the matrix would be denoted by $A^T$. The transpose of a matrix $A$ is found by taking the first row of $A$ and writing it as the first column of $A^T$, then taking the second row of $A$ and writing it as the second column of $A^T$. Continue this process until you have converted each row of $A$ to a column for matrix $A^T$. Below are a couple examples of finding the transpose of a matrix.

<table>
<thead>
<tr>
<th>Matrix $A$</th>
<th>Transpose of $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = \begin{bmatrix} 1 &amp; 3 &amp; 15 \ 2 &amp; 1 &amp; 20 \ 5 &amp; 2 &amp; 1 \end{bmatrix}$</td>
<td>$A^T = \begin{bmatrix} 1 &amp; 2 &amp; 5 \ 3 &amp; 1 &amp; 2 \ 15 &amp; 20 &amp; 1 \end{bmatrix}$</td>
</tr>
<tr>
<td>$A = \begin{bmatrix} -2 &amp; -4 &amp; -24 \ -3 &amp; -3 &amp; -21 \ 4 &amp; 2 &amp; 20 \ 3 &amp; 4 &amp; 1 \end{bmatrix}$</td>
<td>$A^T = \begin{bmatrix} -2 &amp; -3 &amp; 4 &amp; 3 \ -4 &amp; -3 &amp; 2 &amp; 4 \ -24 &amp; -21 &amp; 20 &amp; 1 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Note: $(A^T)^T = A$
The Dual Problem: Every standard minimization model can be written as a standard maximization model. This standard maximization problem is called the dual problem. The optimal solution of the two models are the same value. When I say same optimal solution, I am only referring to the value of the objective function. In other words, the minimum of a standard minimization problem and the maximum of the corresponding dual problem are the same. The values of the decision variables are not the same values.

Setting Up The Dual Problem: We will look at the standard minimization model below.

\[
\begin{align*}
\text{Minimize:} & \quad C = 5x_1 + 2x_2 \\
& \quad x_1 + 3x_2 \geq 15 \\
\text{Subject To:} & \quad \begin{cases} 
2x_1 + x_2 \geq 20 \\
x_1, x_2 \geq 0
\end{cases}
\end{align*}
\]

Notice that I can rewrite the objective function in a different order and list it at the bottom. It is still the same equation and inequalities.

\[x_1 + 3x_2 \geq 15 \quad 2x_1 + x_2 \geq 20 \quad 5x_1 + 2x_2 = 1C\]

(You do not really enter the letter “C”. I just entered it to show you where the 1 came from.)

Now I will write the coefficients from the system in an augmented matrix

\[
A = \begin{bmatrix}
x_1 & x_2 \\
1 & 3 & 15 \\
2 & 1 & 20 \\
5 & 2 & 1C
\end{bmatrix}
\]

Now we get the transpose of the matrix A. Note that I changed the variables from \(x\)’s to \(y\)’s.

\[
A^T = \begin{bmatrix}
y_1 & y_2 \\
1 & 2 & 5 \\
3 & 1 & 2 \\
15 & 20 & 1P
\end{bmatrix}
\]

Again, we do not need to enter the “P” in the transpose. It just shows you, that \(P\) will be the objective function in our dual.

Now we can get the dual problem from the transpose.

\[
\begin{align*}
\text{MAXimize:} & \quad P = 15y_1 + 20y_2 \\
\text{Subject To:} & \quad \begin{cases} 
3y_1 + y_2 \leq 2 \\
y_1, y_2 \geq 0
\end{cases}
\end{align*}
\]

Remember: The dual must be a standard maximization problem, so the objective function must be maximized and the inequalities must be of the form \(\leq\). Also, don’t forget to use different variables than the original problem.

We now can solve the Dual using the simplex method because it is a standard maximization problem. The silver lining is that we already know how to do this.

Steps For Solving A Standard Minimization Problem
1. Write the coefficient matrix \(A\) from the coefficients of the original problem.
2. Transpose the matrix \(A\) and change the variables.
3. Write the dual problem from the transposed matrix.
4. Apply the simplex method to the dual.
5. Interpret the correct solution for the original minimization problem.
Solving The Dual Problem By Simplex Method

**MAXimize**: \( P = 15y_1 + 20y_2 \)

**DUAL PROBLEM**

Subject To: \( 3y_1 + y_2 \leq 2 \)

\( y_1, y_2 \geq 0 \)

When writing the initial system of equations, the \( x \) variables will enter the problem as slack variables.

<table>
<thead>
<tr>
<th>Initial System Of Equations</th>
<th>Initial Simplex Tableau</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 + 2y_2 + x_1 = 5 )</td>
<td>( y_1 \quad y_2 \quad x_1 \quad x_2 \quad P )</td>
</tr>
<tr>
<td>( 3y_1 + y_2 + x_2 = 2 )</td>
<td>( 1 \quad 2 \quad 1 \quad 0 \quad 0 \quad 5 )</td>
</tr>
<tr>
<td>( -15y_1 - 20y_2 + P = 0 )</td>
<td>( 3 \quad 1 \quad 0 \quad 1 \quad 2 )</td>
</tr>
<tr>
<td>( -15 \quad -20 \quad 0 \quad 0 \quad 1 \quad 0 )</td>
<td></td>
</tr>
</tbody>
</table>

Word of caution: As I solve these dual problems, I will not keep track of the entering \( y \) variables. I do not care what these \( y \)-values are. I am interested only in the \( x \)-values and the objective function value.

Where are the values of the decision variables of the original minimization problem found?

Look at the bottom of the \( x_1 \) and \( x_2 \) columns: \( x_1 = 0, x_2 = 20 \)

The best way to explain this is to think about how we got the dual. We kind of turned the problem sideways didn’t we? We rotated it clockwise 90 degrees. If you take the final tableau above and rotate it back counter clockwise, then the \( x \) columns would be rows. And guess what numbers would be to the far right of those rows: \( x_1 = 0, x_2 = 20 \)

That is the best explanation I can give you without confusing you more. At any rate, we have the optimal solution:

**Solution**: Minimum \( C = 40 \) at \( x_1 = 0, x_2 = 20 \)

Next, I will show you from the graphs of the original problem how the minimum and maximums are the same while the decision variables are not.
A Look At The Graphs: We cannot do this for all standard minimization problems because even if we might have only two variables in the minimization problem, we might have more than two variables in the dual problem.

<table>
<thead>
<tr>
<th>Original Problem</th>
<th>Dual Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimize: ( C = 5x_1 + 2x_2 )</td>
<td>Maximize: ( P = 15y_1 + 20y_2 )</td>
</tr>
<tr>
<td>( x_1 + 3x_2 \geq 15 )</td>
<td>( y_1 + 2y_2 \leq 5 )</td>
</tr>
<tr>
<td>Subject To: ( 2x_1 + x_2 \geq 20 )</td>
<td>Subject To: ( 3y_1 + y_2 \leq 2 )</td>
</tr>
<tr>
<td>( x_1, x_2 \geq 0 )</td>
<td>( y_1, y_2 \geq 0 )</td>
</tr>
</tbody>
</table>

Graph of the original problem: The feasible region is the dark blue region farthest out.

Graph of the dual problem: The feasible region is the small triangle near the origin.

Conclusion:
- \((0,20)\) \( C=40 \)
- \((9,2)\) \( C=49 \)
- \((15,0)\) \( C=75 \)

Optimal Solution: Minimum \( C=40 \) at \((0,20)\)

Conclusion:
- \((0,2)\) \( P=40 \)
- \((0,0)\) \( P=0 \)
- \((2/3,0)\) \( P=10 \)

Optimal Solution: Maximum \( P=40 \) at \((0,2)\)

To summarize: The optimal solution to our original minimization problem is \( C=20 \) at \( x_1=0, x_2=20 \). Which is the same minimum as was the maximum to the dual \( (P=40) \)

On To The Dual Method: That’s enough explaining. From this point on I will just solve the standard maximization problems using the dual method.

Note: On the first example we had the same number of variables in the original problem and the dual. That can be deceiving. The number of variables in the dual is not dependent on the number of variables in the original problem. It actually is determined by the number of problem constraints in the original problem. For example: if the original problem has 3 problem constraints, then the dual problem will have 3 decision variables.
Example 1: 
Minimize: \( C = 40x_1 + 12x_2 + 40x_3 \) 
2\(x_1 + x_2 + 5x_3 \geq 20 \) 

Subject To: \( 4x_1 + x_2 + x_3 \geq 30 \) 
\( x_1, x_2, x_3 \geq 0 \)

1. The Coefficient Matrix 
\[ A = \begin{bmatrix} 2 & 1 & 5 & 20 \\ 4 & 1 & 1 & 30 \\ 40 & 12 & 40 & 1 \end{bmatrix} \]

Common mistake is to change the signs of the bottom row. This is not the tableau. You write the coefficients as they are presented in the objective function.

2. The Transpose 
\[ A^T = \begin{bmatrix} 2 & 4 & 10 \\ 1 & 1 & 12 \\ 5 & 1 & 40 \\ 20 & 30 & 1 \end{bmatrix} \]

3. The DUAL PROBLEM 
Maximize: \( P = 20y_1 + 30y_2 \) 
2\(y_1 + 4y_2 \leq 40 \) 
\( y_1 + y_2 \leq 12 \) 
\( 5y_1 + y_2 \leq 40 \) 
\( y_1, y_2 \geq 0 \)

4. Initial System 
\[ \begin{align*} 
2y_1 + 4y_2 + x_1 &= 40 \\
y_1 + y_2 + x_2 &= 12 \\
5y_1 + y_2 + x_3 &= 40 \\
-20y_1 - 30y_2 + P &= 0 
\end{align*} \]

I did not stagger the values here)

5. Initial Simplex Tableau 
\[ \begin{bmatrix} y_1 & y_2 & x_1 & x_2 & x_3 & P \\ 2 & 4 & 1 & 0 & 0 & 0 & 40 \\ 1 & 1 & 0 & 1 & 0 & 0 & 12 \\ 5 & 1 & 0 & 0 & 1 & 0 & 40 \\ -20 & -30 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

This is where the signs change on the bottom row.

6. The Pivoting Process 
\[ \begin{align*} 
& y_1 & y_2 & x_1 & x_2 & x_3 & P \\ 2 & \langle 4 \rangle & 1 & 0 & 0 & 0 & 40 \\
1 & 1 & 0 & 1 & 0 & 0 & 12 \\
5 & 1 & 0 & 0 & 1 & 0 & 40 \\
-20 & [-30] & 0 & 0 & 0 & 1 & 0 
\end{align*} \]

First Pivot 
\[ \begin{align*} 
& 1/2 & \langle 1 \rangle & 1/4 & 0 & 0 & 0 & 10 \\
1 & 1 & 0 & 1 & 0 & 0 & 12 \\
5 & 1 & 0 & 0 & 1 & 0 & 40 \\
-20 & [-30] & 0 & 0 & 0 & 1 & 0 
\end{align*} \]

\[ \begin{align*} 
- R_1 + R_2 & \rightarrow R_2 \\
- R_1 + R_3 & \rightarrow R_3 \\
30R_1 + R_4 & \rightarrow R_4 
\end{align*} \]
### Second Pivot

\[
\begin{bmatrix}
1/2 & 1 & 1/4 & 0 & 0 & 0 & 10 \\
1/2 & 0 & -1/4 & 1 & 0 & 0 & 2 \\
9/2 & 0 & -1/4 & 0 & 1 & 0 & 30 \\
-5 & 0 & 15/2 & 0 & 0 & 1 & 300
\end{bmatrix}
\]

\[Q = 20\]  \[Q = \langle 4 \rangle\]

\[\begin{bmatrix}
1/2 & 1 & 1/4 & 0 & 0 & 0 & 10 \\
1/2 & 0 & -1/2 & 2 & 0 & 0 & 4 \\
9/2 & 0 & -1/4 & 0 & 1 & 0 & 30 \\
-5 & 0 & 15/2 & 0 & 0 & 1 & 300
\end{bmatrix}\rightarrow \begin{bmatrix}
1/2 & 1 & 1/4 & 0 & 0 & 0 & 10 \\
1/2 & 0 & -1/2 & 2 & 0 & 0 & 4 \\
9/2 & 0 & -1/4 & 0 & 1 & 0 & 30 \\
-5 & 0 & 15/2 & 0 & 0 & 1 & 300
\end{bmatrix}

### Final Tableau

\[
\begin{array}{cccccc|c}
& y_1 & y_2 & x_1 & x_2 & x_3 & P \\
0 & 1 & 1/2 & -1 & 0 & 0 & 8 \\
1 & 0 & -1/2 & 2 & 0 & 0 & 4 \\
0 & 0 & 2 & -9 & 1 & 0 & 12 \\
0 & 0 & 5 & 10 & 0 & 1 & 320
\end{array}
\]

### Optimal Solution

\[x_1 = 5\]

\[x_2 = 10\]

\[x_3 = 0\]

Minimum \(C = 320\)

### 1) Another Example

Minimize \(C = 21x_1 + 50x_2\)

\[2x_1 + 5x_2 \geq 12\]

Subject To:

\[3x_1 + 7x_2 \geq 17\]

\[x_1, x_2 \geq 0\]

### 2) The Coefficient Matrix

\[
A = \begin{bmatrix}
2 & 5 & 12 \\
3 & 7 & 17 \\
21 & 50 & 1
\end{bmatrix}
\]

### 3) The Transpose

\[
A^T = \begin{bmatrix}
y_1 & y_2 \\
2 & 3 & 21 \\
5 & 7 & 50 \\
12 & 17 & 1
\end{bmatrix}
\]

### 4) The Dual

Maximize \(P = 12y_1 + 17y_2\)

\[2y_1 + 3y_2 \leq 21\]

Subject To:

\[5y_1 + 7y_2 \leq 50\]

\[y_1, y_2 \geq 0\]

### 5) The System

\[2y_1 + 3y_2 + x_1 = 21\]

\[5y_1 + 7y_2 + x_2 = 50\]

\[-12y_1 - 17y_2 + P = 0\]

### 6) Pivoting

\[
\begin{bmatrix}
2 & \langle 3 \rangle & 1 & 0 & 0 & 21 \\
5 & 7 & 0 & 1 & 0 & 50 \\
-12 & \langle -17 \rangle & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_1 & y_2 & x_1 & x_2 & P \\
2/3 & 1 & 1/3 & 0 & 0 & 7 \\
(1/3) & 0 & -7/3 & 1 & 0 & 1 \\
\langle -2/3 \rangle & 0 & 17/3 & 0 & 1 & 119
\end{bmatrix}
\]

### 7) Final Tableau

\[
\begin{array}{cccccc|c}
& y_1 & y_2 & x_1 & x_2 & P \\
0 & 1 & 1/2 & -1 & 0 & 0 & 8 \\
1 & 0 & -1/2 & 2 & 0 & 0 & 4 \\
0 & 0 & 2 & -9 & 1 & 0 & 12 \\
0 & 0 & 5 & 10 & 0 & 1 & 320
\end{array}
\]

### 8) Optimal Solution

Minimum \(C = 121\)

\[x_1 = 1\]

\[x_2 = 2\]
**Caution:** For the row associated with the objective function, copy the coefficients exactly as written. (This is not where we change their signs.)
You derive the dual problem from the transpose matrix in reverse order, from the way you got the coefficient matrix.
**Caution:** When writing the dual problem, be sure to do the following:
1) Change Minimize to Maximize and change C to P.
2) Change the x-variables to y-variables.
3) Change ≤ constraints to ≥ constraints.
4) Don’t forget the nonnegative constraints.

### Dual Example

**Minimize** : \( C = 16x_1 + 8x_2 + 4x_3 \)

\[
\begin{align*}
3x_1 + 2x_2 + 2x_3 & \geq 16 \\
4x_1 + 3x_2 + x_3 & \geq 14 \\
5x_1 + 3x_2 + x_3 & \geq 12 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

**ST :**

\[
\begin{align*}
y_1, y_2, y_3, x_1, x_2, x_3 & \geq 0
\end{align*}
\]

**The Coefficient Matrix and Transpose**

\[
A = \begin{bmatrix}
3 & 2 & 2 & 16 \\
4 & 3 & 1 & 14 \\
5 & 3 & 1 & 12 \\
16 & 8 & 4 & 1
\end{bmatrix}
A^T = \begin{bmatrix}
3 & 4 & 5 & 16 \\
2 & 3 & 3 & 8 \\
2 & 1 & 1 & 4 \\
16 & 14 & 12 & 1
\end{bmatrix}
\]

**The Dual**

**Maximize** : \( P = 16y_1 + 14y_2 + 12y_3 \)

\[
\begin{align*}
3y_1 + 4y_2 + 5y_3 & \leq 16 \\
2y_1 + 3y_2 + 3y_3 & \leq 8 \\
2y_1 + y_2 + y_3 & \leq 4 \\
y_1, y_2, y_3 & \geq 0
\end{align*}
\]

**ST :**

\[
\begin{align*}
y_1, y_2, y_3, x_1, x_2, x_3 & \geq 0
\end{align*}
\]

**Initial System**

\[
\begin{align*}
y_1 & = 3 \\
y_2 & = 2 \\
y_3 & = 1
\end{align*}
\]

**Initial Tableau:**

\[
\begin{bmatrix}
y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & P \\
3 & 4 & 5 & 1 & 0 & 0 & 0 \\
2 & 3 & 3 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 4 \\
-16 & -14 & -12 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

**Final Tableau:**

\[
\begin{bmatrix}
y_1 & y_2 & y_3 & x_1 & x_2 & x_3 & P \\
0 & 0 & 1 & 1 & -5/2 & -1/4 & 0 \\
0 & 1 & 0 & 1/2 & -1/2 & 0 & 2 \\
1 & 0 & 0 & -1 & 3/4 & 0 & 1
\end{bmatrix}
\]

**Optimal Solution** (for the original minimization problem):

Minimum: \( C=44, \ x_1=0, \ x_2=3, \ x_3=5 \) (found on the bottom row)
Application: We solved this one earlier by the graphing method. The solution was $x_1 = 1$, $x_2 = 6$ and $C = 9.50$. This time we will use the dual method.

A manufacturer of dogfood makes two secret ingredients that goes into their dogfood, codenamed: Superdog and Underdog. Each kg of Superdog contains 300 grams of vitamins, 400 grams of protein, and 100 grams of carbohydrate. Each kg of Underdog contains 100 grams of vitamins, 300 grams of protein, and 200 grams of carbohydrate. Minimum nutritional guidelines require that a mixture made from these ingredients contain at least 900 grams of vitamins, 2200 grams of protein, and 800 grams of carbohydrate. Superdog costs $2.00 per kg to produce and Underdog costs $1.25 per kg to produce. Find the number of kilograms of each ingredient that should be produced in order to minimize cost. (Let $x_1$ = the # of kg of Superdog, and $x_2$ = the # of kg of Underdog)

### The LP Model

Minimize: $C = 2x_1 + 1.25x_2$

Subject to:

$$300x_1 + 100x_2 \geq 900 \quad \text{vitamins}$$
$$400x_1 + 300x_2 \geq 2200 \quad \text{protein}$$
$$100x_1 + 200x_2 \geq 800 \quad \text{carbs}$$
$$x_1, x_2 \geq 0$$

### The Dual Problem

Maximize: $P = 900y_1 + 2200y_2 + 800y_3$

Subject to:

$$300y_1 + 400y_2 + 100y_3 \leq 2$$
$$100y_1 + 300y_2 + 200y_3 \leq 1.25$$
$$y_1, y_2, y_3 \leq 1$$

### Initial System

$$300y_1 + 400y_2 + 100y_3 + x_1 = 2$$
$$100y_1 + 300y_2 + 200y_3 + x_2 = 1.25$$
$$-900y_1 - 2200y_2 - 800y_3 + P = 0$$

### Initial Simplex Tableau

$$\begin{bmatrix}
300 & 400 & 100 & 0 & 0 & 2 \\
100 & 300 & 200 & 0 & 1 & 1.25 \\
-900 & -2200 & -800 & 0 & 0 & 1
\end{bmatrix}$$

$Q = 2 / 400 = 0.005$

$Q = 1.25 / 300 = 0.0042$

### Row Operations

$\frac{1}{300} R_2 \rightarrow R_2$ and $-400R_2 + R_1 \rightarrow R_1$

$2200R_2 + R_1 \rightarrow R_1$

$$\begin{bmatrix}
500/3 & 0 & -500/3 & 1 & -4/3 & 0 \\
1/3 & 1 & 2/3 & 0 & 1/300 & 1/240 \\
-500/3 & 0 & 2000/3 & 0 & 22/3 & 1
\end{bmatrix}$$

$Q = (1/3)(500/3) = 0.002$

$Q = (1/240)(1/3) = 0.0125$

### Row Operations

$\frac{3}{500} R_1 \rightarrow R_1$ and $(-1/3)R_1 + R_2 \rightarrow R_2$

$(500/3)R_1 + R_3 \rightarrow R_3$

### Final Tableau

$$\begin{bmatrix}
y_1 & y_2 & y_3 & x_1 & x_2 & P \\
1 & 0 & -1 & 3/500 & -1/125 & 0 \\
0 & 1 & 1 & -1/500 & 3/500 & 0 \\
0 & 0 & 500 & 1 & 6 & 9.50
\end{bmatrix}$$

Optimal Solution

Minimum $C = 9.50$

$x_1 = 1$ kg, $x_2 = 6$ kg
Final Dual Problem: A computer manufacturing company has two assembly plants, plant A and plant B. They also have distribution outlets I and II. Plant A can assemble at most 700 computers per month and plant B can assemble at most 900 computers per month. Outlet I requires at least 500 computers per month and outlet II requires at least 1000 computers per month. Transportation costs for shipping per computer are:
- $6 to ship from plant A to outlet I
- $5 to ship from plant A to outlet II
- $4 to ship from plant B to outlet I
- $8 to ship from plant B to outlet II

How many computers must be shipped to each route to satisfy all requirements and minimize the shipping costs?

Solution

- $x_1$ = # computers shipped from plant A to outlet I
- $x_2$ = # computers shipped from plant A to outlet II
- $x_3$ = # computers shipped from plant B to outlet I
- $x_4$ = # computers shipped from plant B to outlet II

Objective Function: Minimize: $C = 6x_1 + 5x_2 + 4x_3 + 8x_4$

Assembly Constraints
- Plant A: $x_1 + x_2 \leq 700$
- Plant B: $x_3 + x_4 \leq 900$

Demand Constraints
- Outlet I: $x_1 + x_3 \geq 500$
- Outlet II: $x_2 + x_4 \geq 1000$

Linear Programming Model
Minimize: $C = 6x_1 + 5x_2 + 4x_3 + 8x_4$
Subject To
- $x_1 + x_2 \leq 700$
- $x_3 + x_4 \leq 900$
- $x_1 + x_3 \geq 500$
- $x_2 + x_4 \geq 1000$
- $x_1, x_2, x_3, x_4 \geq 0$

In order to use the dual method, all constraints must be $\geq$. We can multiply the first two constraints by -1 in order to get the model in proper form. See the next cell.

Model Written in Standard Form
Minimize: $C = 6x_1 + 5x_2 + 4x_3 + 8x_4$
Subject To
- $-x_1 - x_2 \geq -700$
- $-x_3 - x_4 \geq -900$
- $x_1 + x_3 \geq 500$
- $x_2 + x_4 \geq 1000$
- $x_1, x_2, x_3, x_4 \geq 0$

Now it is in the proper form and we can apply the dual method.

Coefficient Matrix

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & -700 \\ 0 & 0 & -1 & -1 & -900 \\ 1 & 0 & 1 & 0 & 500 \\ 0 & 1 & 0 & 1 & 1000 \\ 6 & 5 & 4 & 8 & 1 \end{bmatrix}$$

The Transpose

$$A^T = \begin{bmatrix} -1 & 0 & 1 & 0 & 6 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & -1 & 1 & 0 & 4 \\ 0 & -1 & 0 & 1 & 8 \\ -700 & -900 & 500 & 1000 & 1 \end{bmatrix}$$
The Dual Problem

Maximize: \[ P = -700y_1 - 900y_2 + 500y_3 + 1000y_4 \]

Subject To

\[
\begin{align*}
- y_1 + y_3 & \leq 6 \\
- y_1 + y_4 & \leq 5 \\
- y_2 + y_3 & \leq 4 \\
- y_2 + y_4 & \leq 8 \\
y_1, y_2, y_3, y_4 & \geq 0
\end{align*}
\]

Initial System Of Equations

\[
\begin{align*}
y_1 + y_3 + x_1 & = 6 \\
y_1 + y_4 + x_2 & = 5 \\
y_2 + y_3 + x_3 & = 4 \\
y_2 + y_4 + x_4 & = 8 \\
700y_1 + 900y_2 - 500y_3 - 1000y_4 + P & = 0
\end{align*}
\]

Initial Simplex Tableau

\[
\begin{bmatrix}
y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P \\
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\
-1 & 0 & 0 & 0 & \left\{1\right\} & 0 & 1 & 0 & 0 & 5 \\
0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 4 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 8 \\
700 & 900 & -500 & -1000 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

(You can align them better if you like to like the picture below)

Introduce slack variables \( x_1, x_2, x_3, \) and \( x_4, \) and form the initial system for the dual:

\[
\begin{align*}
-y_1 & + \ y_3 & + \ x_1 & = 6 \\
-y_1 & + \ y_4 & + \ x_2 & = 5 \\
- \ y_2 & + \ y_3 & + \ x_3 & = 4 \\
- \ y_2 & + \ y_4 & + \ x_4 & = 8 \\
700y_1 + 900y_2 - 500y_3 - 1000y_4 & + P = 0
\end{align*}
\]

The Pivoting Process

Initial Tableau

\[
\begin{bmatrix}
y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P \\
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\
-1 & 0 & 0 & 0 & \left\{1\right\} & 0 & 1 & 0 & 0 & 5 \\
0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 4 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 8 \\
700 & 900 & -500 & -1000 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\textbf{Row Operations} (pivot element is highlighted)

\(-R_2+R_4\rightarrow R_4\)

\(1000R_2+R_5\rightarrow R_5\)

Next Tableau

\[
\begin{bmatrix}
y_1 & y_2 & y_3 & y_4 & x_1 & x_2 & x_3 & x_4 & P \\
-1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 6 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 \\
0 & -1 & \left\{1\right\} & 0 & 0 & 1 & 0 & 0 & 4 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 3 \\
300 & 900 & -500 & 0 & 0 & 1000 & 0 & 0 & 1 & 5000
\end{bmatrix}
\]

\textbf{Row Operations} (pivot element is highlighted)

\(-R_3+R_1\rightarrow R_1\)

\(500R_1+R_5\rightarrow R_5\)
Subject

Minimize
Maximize

Minimize
Maximize

Final Tableau

Optimal Solution
See Bottom Row

Minimum Cost=$7900

x_1=0, x_2=700, x_3=500, x_4=300

Rewriting a linear programming problem so that it is in standard maximization or standard minimization form.

When a problem is not in one of standard forms, you may be able to rewrite it by multiplying all of the terms of the inequality by −1. {This reverses the direction of the inequality}

Example:

Maximize : \( P = 3x_1 + 5x_2 \)
\( 2x_1 - x_2 \geq -10 \)
Subject To : \( x_1 + 2x_2 \leq 8 \)
\( x_1, x_2 \geq 0 \)

Rewritten:

Maximize : \( P = 3x_1 + 5x_2 \)
\( -2x_1 + x_2 \leq 10 \)
Subject To : \( x_1 + 2x_2 \leq 8 \)
\( x_1, x_2 \geq 0 \)

Example:

Minimize : \( C = 21x_1 + 50x_2 \)
\( x_1 - 5x_2 \leq 2 \)
Subject To : \( 3x_1 + 7x_2 \geq 17 \)
\( x_1, x_2 \geq 0 \)

Rewritten:

Minimize : \( C = 21x_1 + 50x_2 \)
\( -x_1 + 5x_2 \geq -2 \)
Subject To : \( 3x_1 + 7x_2 \geq 17 \)
\( x_1, x_2 \geq 0 \)

The following problem can’t be “fixed”.

Maximize : \( P = 3x_1 + 5x_2 \)
\( 2x_1 - x_2 \geq 10 \)
Subject To : \( x_1 + 2x_2 \leq 8 \)
\( x_1, x_2 \geq 0 \)

Maximize : \( P = 3x_1 + 5x_2 \)
\( -2x_1 + x_2 \leq -10 \)
Subject To : \( x_1 + 2x_2 \leq 8 \)
\( x_1, x_2 \geq 0 \)

Even though you can fix the inequality; in standard max problems the constants can’t be negative. This example is what we call a mixed constraint problem. Mixed constraint problems are discussed in the next section.